Bosonization of fermion determinants

A.A.Slavnov [*]

Steklov Mathematical Institute, Russian Academy of Sciences,
Vavilov st.42, GSP-1,117966, Moscow, Russia

February 1, 2008

Abstract

A four dimensional fermion determinant is presented as a path integral of the exponent of a local five dimensional action describing constrained bosonic system. The construction is carried out both in the continuum theory and in the lattice model.

1 Introduction

The problem of bosonization of fermionic theories was studied by many authors (see e.g. [1], [2], [3], [4], [5], [6]), mainly in the context of two-dimensional models. There were also some attempts to extend this procedure to higher dimensions [7], [8], [9], [10], but at present they lead only to partial success.

The problem of bosonization is of a particular importance for lattice gauge theories as lattice QCD, because the fermionic degrees of freedom complicate numerical simulations enormously.

Recently M.Lusher [11] proposed the algorithm for the approximate inversion of the QCD fermion determinant replacing it by an infinite series of bosonic determinants. In the present paper I will describe an alternative approach which allows to write the exact expression for the fermion determinant as a path integral of the exponent of a local bosonic action. In my approach a four dimensional fermionic system is replaced by a five dimensional constrained bosonic one.

In the second section I present the construction for the continuum theory. In this case the proof of equivalence will be given in the framework of perturbation theory as the problem of nonperturbative definition of a continuum path integral is at present beyond our possibilities. In the third section the corresponding procedure will be given for the lattice QCD where the equivalence may be proven at the nonperturbative level as well.

[*]E-mail: slavnov@class.mian.su

2 Perturbative bosonization in the continuum theory

We consider the following fermion determinant

$$\det(D+m)^n = \int \exp\{-\sum_{i=1}^{n}\int d^4x[\bar{\psi}^i_\alpha(x)(D_{\alpha\beta}+m\delta_{\alpha\beta})\psi^i_\beta(x)]\}d\bar{\psi}^i_\alpha d\psi^i_\beta \qquad (1)$$

where

$$D_{\alpha\beta} = (\gamma_\mu\partial_\mu + ig\gamma_\mu A_\mu)_{\alpha\beta} \qquad (2)$$

A_μ belongs to the Lie algebra of some compact group ($SU(3)$ for QCD). Indices α denote spinorial and colour components. The eq.(1) describes n degenerate fermion flavours interacting vectorialy with the Yang-Mills field. To make contact with the discussion of lattice models we assume that the four dimensional space is Euclidean and the γ_μ matrices ($\mu = 1, 2 \ldots 5$) are Hermitean. As all the discussion in this section will be restricted to perturbation theory all the equations are understood as a series over g, and the Gaussian path integrals are defined axiomatically as in ref. [12], [13]. Therefore we need not to bother about the convergence of the path integrals and all the conclusions are valid (perturbatively) for any number of fermion flavours. However in the next section where we deal with the nonperturbative proof for lattice models we have to be more careful at this point and to make all the integrals convergent we restrict our discussion to the case of even number of flavours. For simplicity we take $n = 2$.

We present the determinant of the Dirac operator as the determinant of the Hermitean operator by using the identity

$$\det(D+m) = \det[\gamma_5(D+m)] \qquad (3)$$

Therefore

$$\det(D+m)^2 = \det(\gamma_5(D+m)\gamma_5(D+m)) = \det(-D^2 + m^2) \qquad (4)$$

Being the square of the Hermitean operator the operator $-D^2 + m^2$ is positive definite.

Let us introduce the following action depending on five dimensional bosonic fields $\phi_\alpha(t, x)$

$$S = \int dt d^4x\{\lambda\phi^*_\alpha(t,x)\partial_t\phi_\alpha(t,x) + \phi^*_\alpha(t,x)[-D^2(x)+m^2]_{\alpha\beta}\phi_\beta(t,x)\} \qquad (5)$$

Here λ is a small positive parameter which eventually will be put equal to zero. The fields ϕ_α have the same spinorial and colour structure as the fields ψ_α.

We claim that the determinant (1) (for $n = 2$) may be presented as the following path integral:

$$\det(D+m)^2 = \lim_{\lambda\to 0}\int \exp\{-S + i\int_0^1 dt\int d^4x\phi^*_\alpha(x,t)\chi_\alpha(x) +$$

$$+i\int_0^1 dt\int d^4x\chi^*_\alpha(x)\phi_\alpha(x,t)\}d\phi^*_\alpha d\phi_\beta d\chi^*_\alpha d\chi_\beta \qquad (6)$$

The fields $\chi(x)$ play a role of Lagrange multipliers imposing the constraints

$$\int_0^1 dt\phi^*(t,x) = \int_0^1 dt\phi(t,x) = 0 \tag{7}$$

The Green functions of the fields ϕ defined by the action(5) are retarded:

$$\tilde{G}(k,p) = \frac{1}{i\lambda k + p^2 + m^2} \tag{8}$$

$$G(t,p) = \frac{1}{2\pi}\int \exp\{ikt\}\tilde{G}(k,p)dk; \quad G(t) = 0, t < 0 \tag{9}$$

Integrating in the eq.(6) over ϕ^*, ϕ one gets

$$I = \det(\lambda\partial_t - D^2 + m^2)^{-1} \times \tag{10}$$

$$\int \exp\{-\int_0^1 dtds \int d^4x d^4y \chi_\alpha^*(x)[\lambda\partial_t - D^2 + m^2]_{\alpha\beta}^{-1}(x,y,s-t)\chi_\beta(y)\}d\chi^*d\chi$$

Due to the fact that the Green functions of the fields ϕ are retarded, for any λ different from zero the first factor in the eq.(10) is equal to one. Indeed a perturbative expansion of this determinant generates the diagrams of the type

$$\Pi_n \sim \int dk \frac{F(k,p_1\dots p_n)}{(i\lambda k + p_1^2 + m^2)\dots(i\lambda k + p_n^2 + m^2)} \tag{11}$$

where F is some polynomial over p_i, k. Note that the fields A do not depend on s, therefore the diagrams generated by eq.(10) are the cycles with zero fifth component of momenta. The integral over k is convergent and all the poles are in the upper complex half plane. Closing the integration contour in the lower half plane one sees that the integral is equal to zero. In fact one has to be slightly more careful at this point, as due to momentum conservation all these diagrams are multiplied by the singular constant factor $\sim \delta(0)$. To give a precise meaning to this expression some infrared cut-off must be introduced to the eq. (5). In the next section we shall show that a finite lattice may serve this purpose.

The second factor is not singular in the limit $\lambda \to 0$ and one can calculate it by putting $\lambda = 0$. Let us prove it in some more details.

Expansion of the exponent in the eq.(10) over g generates the terms of the form

$$\int_0^1 ds \int_0^1 dt \int dk \frac{\exp\{ik(s-t)\}P(\lambda k, p_1\dots p_n)}{(i\lambda k + p_1^2 + m^2)\dots(i\lambda k + p_n^2 + m^2)} \tag{12}$$

where P is some polynomial over k, p_i. Integrating over s and t one gets

$$\int dk \frac{(2 - e^{-ik} - e^{ik})P(\lambda k, p_i)}{(k - i\epsilon)^2(i\lambda k + p_1^2 + m^2)\dots(i\lambda k + p_n^2 + m^2)} \tag{13}$$

The pole at $k = 0$ is spurious. It is compensated by the zero of the nominator. So we are free to define the roundabout of this pole arbitrarily. We choose the prescription $k \to k - i\epsilon$. With this prescription the part of the integrand proportional

3

to $(2-\exp\{-ik\})$ gives zero contribution as in this case we may close the integration contour in the lower half plane and all the poles are situated in the upper half plane. The last term, proportional to $\exp\{ik\}$ is calculated by closing the contour in the upper half plane. The integral is equal to the sum of the residues at the poles at $k = 0, k = i\lambda^{-1}(p_i^2 + m^2)$. The contribution of the poles at $k = i\lambda^{-1}(p_i^2 + m^2)$ vanishes at $\lambda \to 0$. The contribution of the pole at $k = 0$ gives

$$\frac{P(k, p_i)}{(i\lambda k + p_1^2 + m^2)\ldots(i\lambda k + p_n^2 + m^2)}|_{k=0} \tag{14}$$

Therefore the integral (10) acquires a form

$$I = \int \exp\{-\int \chi_\alpha^*(x)[-D^2 + m^2]_{\alpha\beta}^{-1}(x, y)\chi_\beta(y)dxdy + O(\lambda)\}d\chi_\alpha^* d\chi_\beta \tag{15}$$

Integrating over χ^*, χ we get

$$I = \det(-D^2 + m^2) + O(\lambda) \tag{16}$$

Therefore

$$\lim_{\lambda \to 0} I = \det(D + m)^2 \tag{17}$$

3   Bosonization on the lattice

In this section we generalize the construction given above to the case of lattice models, where the proof of equivalence can be given without any references to perturbation theory.

We again present the determinant of the Dirac operator as the determinant of the Hermitean operator by using the identity

$$\det(\hat{D} + m) = \det[\gamma_5(\hat{D} + m)], \quad \hat{D} = \gamma_\mu D_\mu \tag{18}$$

In eq.(18) D_μ is the lattice covariant derivative

$$D_\mu \psi(x) = \frac{1}{2a}[U_\mu^+(x)\psi(x + a_\mu) - U_\mu(x)\psi(x - a_\mu] \tag{19}$$

U_μ is a lattice gauge field. We consider a finite lattice with periodic boundary conditions.

As it has been already mentioned, to provide the positivity of the determinant we consider the case of two degenerate fermion flavours interacting vectorialy with the Yang-Mills field. We shall not deal explicitely with the problem of fermion doubling. All the reasonings are trivially extended to the model improved by adding for example the Wilson term.

It is convinient to present the fermion determinant in the following form

$$\det[\gamma_5(\hat{D} + m)]^2 = \int \exp\{a^4 \sum_x \bar{\psi}(x)(\hat{D}^2 - m^2)\psi(x)\}d\bar{\psi}d\psi \tag{20}$$

4

We shall prove that in analogy with the continuum case the integral over fermionic fields ψ can be replaced by an integral of a five dimensional lattice bosonic action. The spatial components x are defined as above. The fifth component t to be defined on the one dimensional lattice of the length L with the lattice spacing b:

$$L = 2Nb, \quad -N < n \leq N \tag{21}$$

We choose b in such a way that $b << a$ and in the continuum limit

$$2Nb^2 = Lb \to 0 \tag{22}$$

The key ingredient of the construction presented in the preceeding section was the using of retarded Green functions for the fields ϕ providing triviality of the determinant in the eq.(10). This phenomenon is a continuum analog of the well known property of a triangular matrix: the determinant of a triangular matrix is equal to the product of the diagonal elements. Therefore to achieve the same goal in the lattice case we choose the lattice derivative with respect to the fifth coordinate t in the form of a triangular matrix. More precisely we are going to prove the following equality

$$\int \exp\{a^4 \sum_x \bar{\psi}(x)(\hat{D}^2 - m^2)\psi(x)\}d\bar{\psi}d\psi = \tag{23}$$

$$= \lim_{\lambda \to 0, b \to 0} \int \exp\{a^4 b \sum_{n=-N+1}^{N} \sum_x [\lambda \frac{\phi_{n+1}^*(x) - \phi_n^*(x)}{b}\phi_n(x) + \phi_n^*(x)(\hat{D}^2 - m^2)\phi_n(x) -$$

$$- \frac{i}{\sqrt{L}}(\phi_n^*(x)\chi(x) + \chi^*(x)\phi_n(x))]\}d\phi_n^*d\phi_n d\chi^* d\chi.$$

Here $\phi_n(x)$ and $\chi(x)$ are bosonic fields which carry the same spinorial and colour indices as the fields $\psi(x)$. The fields $\phi_n(x)$ satisfy free boundary conditions i.e.

$$\phi_n = 0, \quad n \leq -N, \quad n > N \tag{24}$$

The operator $-\hat{D}^2 + m^2$ is Hermitean and can be diagonalized by a unitary transformation. As it does not depend on t this transformation will make the exponent in the r.h.s. of eq.(23) diagonal with respect to all variables except for t. After this transformation the r.h.s. of eq.(23) acquires the form

$$I = \lim_{\lambda \to 0, b \to 0} I(\lambda, b)$$

$$I(\lambda, b) = \int \exp\{b \sum_{n=-N+1}^{N} \sum_\alpha [\lambda \frac{\phi_{n+1}^{*\alpha} - \phi_n^{*\alpha}}{b}\phi_n^\alpha - \phi_n^{*\alpha}B^\alpha \phi_n^\alpha$$

$$- \frac{i}{\sqrt{L}}(\phi_n^{*\alpha}\chi^\alpha + \chi_n^{*\alpha}\phi_n^\alpha)]\}d\phi_n^*d\phi_n d\chi^* d\chi; \quad \phi_{N+1}^{*\alpha} = 0 \tag{25}$$

Index α refers now to the eigenstates of the operator $-\hat{D}^2 + m^2$, B_α being the corresponding eigenvalues. Obviously $B_\alpha > 0$.

5

The integral (25) is convergent as the real part of the action in the exponent is positive. It is easy to see by rewriting the action in terms of Fourier components:

$$S = \frac{1}{2\pi} \int_{-\frac{\pi}{b}}^{\frac{\pi}{b}} \{\tilde{\phi}^{*\alpha}(k)[-\lambda(e^{-ikb} - 1)b^{-1} + B^\alpha]\tilde{\phi}^\alpha(k)] +$$

$$+ \frac{i}{2\pi\sqrt{L}}(e^{-iknb}\tilde{\phi}^{*\alpha}(k)\chi^\alpha + e^{iknb}\chi^{*\alpha}\tilde{\phi}^\alpha(k))\}dk \tag{26}$$

The real part is

$$ReS = \frac{1}{2\pi} \int_{-\frac{\pi}{b}}^{\frac{\pi}{b}} \{\tilde{\phi}^{*\alpha}(k)[-\lambda(\cos kb - 1)b^{-1} + B^\alpha]\tilde{\phi}^\alpha(k)]dk \tag{27}$$

As $\lambda < 0$ and $B^\alpha > 0$, $ReS > 0$.

Performing in the eq.(25) the integration over ϕ_n one gets

$$I(\lambda, b) = \det(C^{-1}) \int \exp\{\frac{-b}{L} \sum_\alpha \sum_{n,m=-N+1}^{N} \chi^{*\alpha}(C^\alpha)_{mn}^{-1}\chi^\alpha\}d\chi^*d\chi \tag{28}$$

Where $C_\alpha(k)$ is the kernel of the quadratic form in the eq.(25).

As follows from the eq.(25)

$$\det(C) = \prod_\alpha \det(C_\alpha) \tag{29}$$

In the coordinate space C_α is a triangular matrix with the diagonal elements

$$\sim (\lambda + B_\alpha b) \tag{30}$$

Therefore

$$\det(C) = \exp\{\sum_\alpha \ln(\lambda + B_\alpha b)^{2N}\} \tag{31}$$

Separating the constant term one gets

$$\det(C) = \exp\{2Nb\lambda^{-1} \sum_\alpha B_\alpha + O(Lb)\} = \tag{32}$$

$$= \exp\{\lambda^{-1}L \operatorname{Tr}[-\hat{D}^2 + m^2] + O(Lb)\}$$

Using the explicit form of the operator D one can easily verify that $\operatorname{Tr}[-\hat{D}^2 + m^2]$ is a nonessential constant. It follows also from the fact that the trace of a local operator is local. It has to be gauge invariant and a polynomial of the second order in the fields A_μ. The only possible solution is a constant. So we can include $\det(C^{-1})$ into normalization constant.

To get an explicit form of the remaining terms it is sufficient to find the stationary point of the exponent in the eq.(25). The corresponding classical equations look as follows

$$- \lambda(\phi_{n+1}^{*\alpha} - \phi_n^{*\alpha})b^{-1} + B^\alpha\phi_n^{*\alpha} + iL^{-\frac{1}{2}}\chi^{*\alpha} = 0; \quad n \neq N \tag{33}$$

$$\lambda(\phi_n^\alpha - \phi_{n-1}^\alpha)b^{-1} + B^\alpha\phi_n^\alpha + iL^{-\frac{1}{2}}\chi^\alpha = 0; \quad n \neq -N+1$$

6

$$-\phi_N^{*\alpha}b^{-1}\lambda - \phi_N^{*\alpha}B^\alpha - iL^{-\frac{1}{2}}\chi^{*\alpha} = 0 \tag{34}$$

$$\phi_{-N+1}^\alpha b^{-1}\lambda + \phi_{-N+1}^\alpha B^\alpha + iL^{-\frac{1}{2}}\chi^\alpha = 0$$

for small b eq.s(33) may be approximated by the differential equations

$$-\lambda\partial_t\phi^{*\alpha} + B^\alpha\phi^{*\alpha} + iL^{-\frac{1}{2}}\chi^{*\alpha} = 0 \tag{35}$$

$$\lambda\partial_t\phi^\alpha + B^\alpha\phi^\alpha + iL^{-\frac{1}{2}}\chi^\alpha = 0$$

whereas eq.s(34) play the role of boundary conditions:

$$\phi^{*\alpha}(\frac{L}{2}) = 0; \quad \phi^\alpha(-\frac{L}{2}) = 0 \tag{36}$$

The solution of these eq.s is

$$\phi^{*\alpha}(t) = -\frac{i}{\sqrt{L}B^\alpha}\chi^{*\alpha}(1 - \exp\{B^\alpha\lambda^{-1}(t - \frac{L}{2})\}) \tag{37}$$

$$\phi^\alpha(t) = -\frac{i}{\sqrt{L}B^\alpha}\chi^\alpha(1 - \exp\{-B^\alpha\lambda^{-1}(t + \frac{L}{2})\})$$

Substituting these solutions to the eq.(25) we get in the limit $b \to 0$

$$\lim_{b\to 0} I(\lambda, b) = \int \exp\{-\int_{-\frac{L}{2}}^{\frac{L}{2}} \sum_\alpha \frac{\chi^{*\alpha}\chi^\alpha}{B^\alpha L}[1 - \exp\{\frac{B^\alpha}{\lambda}(t - \frac{L}{2})\}]dt\}d\chi^*d\chi = \tag{38}$$

$$= \int \exp\{-\sum_\alpha \chi^{*\alpha}(B^\alpha)^{-1}\chi^\alpha[1 - \frac{\lambda}{B^\alpha L}[1 + \exp\{-B^\alpha\lambda^{-1}L\}]]\}d\chi^*d\chi$$

In the limit $\lambda \to 0$ this equation reduces to

$$I = \lim_{\lambda\to 0, b\to 0} I(\lambda, b) = \exp\{-\sum_\alpha \chi^{*\alpha}(B^\alpha)^{-1}\chi^\alpha\}d\chi^{*\alpha}d\chi^\alpha = \det(-\hat{D}^2 + m^2) \tag{39}$$

The equality (23) is proven. The final recipe is the following: Any gauge invariant observable may be expressed as a purely bosonic path integral of the form

$$Z = \lim_{\lambda\to 0, b\to 0} \int \exp\{a^4 b \sum_{n=-N+1}^{N} \sum_x [\lambda\frac{\phi_{n+1}^*(x) - \phi_n^*(x)}{b}\phi_n(x) +$$

$$+\phi_n^*(x)(\hat{D}^2 - m^2)\phi_n(x) - \frac{i}{\sqrt{L}}(\phi_n^*(x)\chi(x) + \chi^*(x)\phi_n(x))] +$$

$$+L(U) + s.t.\}d\phi_n^*d\phi_n d\chi^*d\chi \tag{40}$$

where $L(U)$ is the lattice Yang-Mills action and s.t. stands for the source term depending on the fields U_μ, ϕ_n^*, ϕ_n. All the fields are bosonic and the integration goes over the fields ϕ_n^*, ϕ_n defined on the one-dimensional lattice with length L and free boundary conditions.

4 Discussion

In this paper we showed that a four dimensional fermion determinant can be written as a path integral of the exponent of a five dimensional local bosonic action. In the same way one can present a two dimensional fermion determinant as a path integral for three-dimensional bosonic theory. In the case of lattice models this procedure leads to a well defined bosonic path integral. No numerical simulations in this approach have been tried so far and it would be very important to see how the method works in practical calculations. In this case one need not of course to take the limit $\lambda \to 0, b \to 0$, but the following conditions $b << a, \quad b << \lambda, \quad b << L^{-1}$ must be respected.

References

[1] S.Coleman, Phys.Rev.D11 (1975) 2088.

[2] A.Polyakov and P.Wigman, Phys.Lett.B131 (1983) 121; 141 (1984) 223.

[3] E.Witten, Comm.Math.Phys.92 (1984) 455.

[4] R.Gamboa-Saravi, F.Shaposnik and J.Solomin, Nucl.Phys.B185 (1981) 239.

[5] K.Furuya, R.Gamboa-Saravi and F.Shaposnik, Nucl.Phys.B208 (1982) 159.

[6] C.Naon, Phys.Rev. D31 (1985) 2035.

[7] G.W.Semenoff, Phys.Rev.Lett.61 (1988) 817.

[8] M.Lusher, Nucl.Phys. B326 (1989) 557.

[9] E.C.Marino, Phys.Lett. B263 (1991) 63.

[10] L.Huerta,F.Zanelli, Phys.Rev.Lett. 71 (1993) 3622.

[11] M.Lusher Nucl.Phys. B418 (1994) 637.

[12] A.A.Slavnov, Theor.Math.Phys. 22 (1975) 177.

[13] L.D.Faddeev, A.A.Slavnov,*Gauge Fields. Introduction to Quantum Theory.* Second edition. Moscow, Nauka, 1988. English translation ed.Addison-Wesley 1990